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An Extreme Black Hole with Electric Dipole Moment

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Abstract

We construct a new extreme black hole solution in toroidally compactified heterotic string theory. The black hole saturates the Bogomol'nyi bound, has zero angular momentum, but nonzero electric dipole moment. It is obtained by starting with a higher dimensional rotating charged black hole, and compactifying one direction in the plane of rotation.

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The black hole uniqueness theorems show that any static black hole with a regular event horizon must be spherically symmetric. This implies that all multipole moments vanish except for the mass and charge. Uniqueness theorems have been proven for black holes in vacuum [1], Einstein-Maxwell theory [2], and more recently, Einstein-Maxwell-dilaton theory [3]. However, when the dilaton is present, the event horizon may shrink down to zero size in the extremal limit and become singular. In this case, the theorems no longer apply. Of course any limit of static, nonextremal black holes must remain spherically symmetric. However, it has recently been realized that extreme black holes are qualitatively different objects and may have properties which are not shared by their nonextreme analogs.

In this letter we will construct a static extreme black hole with nonzero electric dipole moment. While the magnitude of the charge is related to the mass by the extremality condition, the dipole moment is arbitrary, and is an independent parameter in the solution. The solution involves more than one $U(1)$ gauge field, and the dipole moment is associated with a different gauge field than the charge.

We will work in the context of heterotic string theory compactified on a torus. Our starting point is the observation [4] that in more than five dimensions, extremally charged rotating black holes saturate a Bogomol'nyi bound. This is in contrast to the situation in lower dimensions where the extremal limit involves the angular momentum and does not saturate this bound. When the Bogomol'nyi bound is saturated, there is no force between the black holes and they can be superposed. By taking an infinite periodic array of these higher dimensional black holes, one can compactify some of the spatial directions and obtain effectively four dimensional solutions. This approach was used in [4] to obtain rotating black holes in four dimensions which saturate the Bogomol'nyi bound. To preserve the angular momentum, it was necessary to compactify directions which were orthogonal to the plane of rotation. We will show that by compactifying a direction in the plane of rotation, one obtains a nonrotating extreme black hole with nonzero dipole moment.

To begin, we review the six-dimensional, extremally charged, rotating black hole solution in heterotic string theory compactified on a four dimensional torus [4]. Since the scalar

moduli fields remain constant for this solution[1], it suffices to work with the following low energy effective action

$$S = \int d^6x \sqrt{-\det G}\, e^{-\Phi} \left[R_G + \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \sum_{j=1}^{4} F^{(j)}_{\mu\nu} F^{\mu\nu(j)} \right] \tag{1}$$

where

$$F^{(j)}_{\mu\nu} = \partial_\mu A^{(j)}_\nu - \partial_\nu A^{(j)}_\mu \,, \tag{2}$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2 \sum_{j=1}^{4} A^{(j)}_\mu F^{(j)}_{\nu\rho} + \text{cyclic permutations of } \mu,\ \nu,\ \rho \,. \tag{3}$$

$A^{(j)}_\mu$ are four $U(1)$ gauge fields that arise from the compactification on T^4, and R_G is the scalar curvature associated with the metric $G_{\mu\nu}$. To describe the solution, it is convenient to introduce ρ and θ which are related to the Cartesian coordinates $x^1, \ldots, x^5$ and a parameter a (which will be related to the angular momentum) as follows;

$$(x^1)^2 + (x^2)^2 = (\rho^2 + a^2) \sin^2\theta, \tag{4}$$

$$R^2 \equiv \sum_{i=1}^{5} (x^i)^2 = \rho^2 + a^2 \sin^2\theta \,. \tag{5}$$

An extremally charged black hole which is rotating in the x^1, x^2 plane can be described in IWP form [5,6] in terms of a function $F(\vec{x})$ and vector $\omega_i(\vec{x})$ as follows [4];

$$ds^2 = -F^2(\vec{x})[dt + \omega_i(\vec{x}) dx^i]^2 + d\vec{x}^2 \,, \tag{6}$$

$$A^{(j)}_t = \frac{p^{(j)}}{\sqrt{2}} [F - 1] \,, \qquad A^{(j)}_i = \frac{p^{(j)}}{\sqrt{2}} F \omega_i \qquad (j = 1, \ldots, 4), \tag{7}$$

$$\Phi = \ln F(\vec{x}) \,, \qquad B_{ti} = -F\omega_i \,, \qquad B_{ik} = 0 \,, \tag{8}$$

where

$$F^{-1} = 1 + \frac{m_0}{\rho(\rho^2 + a^2 \cos^2\theta)} \,, \qquad \omega_i dx^i = \frac{m_0 a (x^1 dx^2 - x^2 dx^1)}{\rho(\rho^2 + a^2\cos^2\theta)(\rho^2 + a^2)}, \tag{9}$$

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[1] We consider black holes with "right moving" charge only. For a more general solution, see [4].

and $p^{(j)}$ is a set of constants which satisfy $\sum_{j=1}^{4}(p^{(j)})^2 = 1$.

Several comments on this solution are in order. The mass, total charge, and angular momentum are given by $M = 3m_0/2$, $Q^2 \equiv \sum_{j=1}^{4}(Q^{(j)})^2 = 9m_0^2/2$, and $J = m_0 a/2$. So $M^2 = Q^2/2$ as expected for an extreme dilatonic black hole. Notice that J is independent of M even though the black hole is extremal. This is related to the fact that rotating uncharged black holes exist in six dimensions for all values of M and J [7]. There is no extremal limit in this case. Another unusual property is that there is no ergosphere; the Killing vector $\partial/\partial t$ is timelike everywhere. The surface $\rho = 0$ is a curvature singularity which is null. This is a result of the event horizon shrinking down to zero size. The spacetime is free of naked (timelike) singularities. Finally, this solution arises from the dimensional reduction of a 'chiral null model' [8] which does not receive α' corrections in a particular renormalization scheme.

For configurations of the form (6 - 8), the field equations derived from (1) reduce to the following linear equations for F^{-1} and ω_i

$$\sum_{i=1}^{5} \partial_i \partial_i F^{-1} = 0, \qquad \sum_{i=1}^{5} \partial_i \partial_{[i} \omega_{j]} = 0, \tag{10}$$

One can thus construct multi-black hole solutions by superposing these single black holes. This is a reflection of the fact that there is no force between objects which saturate a Bogomol'nyi bound. In the following we study black hole solutions obtained by superposing the solution (6-9) and its spatial translation.

If we place an infinite number of black holes along a line a unit distance apart, the resulting space time has a periodicity one along the line. This procedure is equivalent to compactifying one dimension on a torus with the period one [9–12]. By putting six dimensional black holes spinning in the x^1, x^2 plane along the x^4 and x^5 directions one obtains a four-dimensional rotating black hole which saturates the Bogomol'nyi bound without naked singularities [4].

We are interested in compactifying one of the directions in the plane of rotation e.g. the x^1 direction. The asymptotic values of F^{-1} and ω_i for a single six dimensional black hole

are given by[2]

$$F^{-1} \simeq 1 + \frac{m_0}{R^3}\,, \qquad \omega_i dx^i \simeq \frac{m_0 a(x^1 dx^2 - x^2 dx^1)}{R^5}\,. \tag{11}$$

A periodic array of black holes in the x^1 direction with periodicity one then yields the following asymptotic values for F^{-1} and ω_i:

$$\breve{F}^{-1} \simeq 1 + \sum_{n=-\infty}^{\infty} \frac{m_0}{\{\breve{R}^2 + (x^1 - n)^2\}^{\frac{3}{2}}}\,, \tag{12}$$

$$\breve{\omega}_i dx^i \simeq \sum_{n=-\infty}^{\infty} \frac{m_0 a\{(x^1 - n)dx^2 - x^2 dx^1\}}{\{\breve{R}^2 + (x^1 - n)^2\}^{\frac{5}{2}}} \tag{13}$$

where

$$\breve{R}^2 \equiv \sum_{i=2}^{5} (x^i)^2 = R^2 - (x^1)^2. \tag{14}$$

For $\breve{R} \gg 1$ the summations on the right hand side of (12) and (13) can be evaluated by integration as follows[3]

$$\breve{F}^{-1} \simeq 1 + \int_{-\infty}^{\infty} du \frac{m_0}{\{\breve{R}^2 + (x^1 - u)^2\}^{\frac{3}{2}}}\,, \tag{15}$$

$$\breve{\omega}_1 \simeq -\int_{-\infty}^{\infty} du \frac{m_0 a x^2}{\{\breve{R}^2 + (x^1 - u)^2\}^{\frac{5}{2}}} \quad , \qquad \breve{\omega}_2 \simeq \int_{-\infty}^{\infty} du \frac{m_0 a (x^1 - u)}{\{\breve{R}^2 + (x^1 - u)^2\}^{\frac{5}{2}}}\,. \tag{16}$$

By changing variable $u \equiv x^1 + \breve{R}v$ one can easily perform the integrals above and obtain

$$\breve{F}^{-1} \simeq 1 + \frac{m_0}{\breve{R}^2} \int_{-\infty}^{\infty} dv \frac{1}{(1+v^2)^{\frac{3}{2}}} = 1 + \frac{2m_0}{\breve{R}^2}, \tag{17}$$

$$\breve{\omega}_1 \simeq -\frac{m_0 a}{\breve{R}^4} x^2 \int_{-\infty}^{\infty} dv \frac{1}{(1+v^2)^{\frac{5}{2}}} = -\frac{4 m_0 a}{3\breve{R}^4} x^2\,, \tag{18}$$

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[2]Expanding to higher order, one finds that F^{-1} has a quadrupole moment term $m_0 a^2 \left(\frac{3}{2}\sum_{i=1}^{2}(x^i)^2 - \sum_{i=3}^{5}(x^i)^2\right)/R^7$ but no dipole moment.

[3]Similar sums can be evaluated analytically with the result that the exact answer differs from the leading term computed here only by exponentially small $O(\exp(-\breve{R}))$ contributions. (See e.g. [9].)

$$\breve{\omega}_2 \simeq -\frac{m_0 a}{\breve{R}^3} \int_{-\infty}^{\infty} dv \frac{v}{(1+v^2)^{\frac{5}{2}}} = 0. \tag{19}$$

To proceed, there are essentially two choices for the further compactification direction. If we compactify the second direction in the rotation plane, x^2, with the same procedure we find

$$\hat{F}^{-1} \equiv 1 + \sum_{n=-\infty}^{\infty} [\breve{F}^{-1}(t, x^1, x^2 - n, x^3, x^4, x^5) - 1] \simeq 1 + \frac{2\pi m_0}{\hat{R}}\,, \tag{20}$$

$$\hat{\omega}_i \equiv \sum_{n=-\infty}^{\infty} \breve{\omega}_i(t, x^1, x^2 - n, x^3, x^4, x^5) \simeq 0\,, \tag{21}$$

where

$$\hat{R}^2 \equiv \sum_{i=3}^{5} (x^i)^2. \tag{22}$$

In this case, the asymptotic geometry is that of the standard extremally charged dilatonic black hole [13,14]

$$ds^2 = -\left(1 + \frac{2\pi m_0}{\hat{R}}\right)^{-2} dt^2 + \sum_{i=3}^{5} (dx^i)^2\,, \tag{23}$$

$$\Phi = -\ln\left(1 + \frac{2\pi m_0}{\hat{R}}\right)\,, \tag{24}$$

$$A_t^{(j)} = \frac{p^{(j)}}{\sqrt{2}} \left[\left(1 + \frac{2\pi m_0}{\hat{R}}\right)^{-1} - 1\right]\,, \tag{25}$$

$$A_i^{(j)} = 0 \qquad\qquad (j = 1, \ldots, 4)\,. \tag{26}$$

All other fields vanish. This includes the four-dimensional components of the anti-symmetric tensor $B_{\mu\nu}$, and additional $U(1)$ gauge fields and scalars that could arise in the compactification down to four dimensions. The above expressions for the geometry, $U(1)$ gauge fields and scalar dilaton agree with the standard four-dimensional charged dilatonic black hole with mass $M = \pi m_0$ and total charge $Q^2 = 2\pi^2 m_0^2$. Although we derived (23-26) keeping only the leading order terms for large $\hat{R}$, it nevertheless accurately describes the four dimensional

solution down to the compactification scale. This is because the original solution treated the coordinates x^i, $i = 3, 4, 5$ symmetrically, so higher order terms in the expansion of F and ω_i cannot lead to higher order multipole moments.

The more interesting alternative is to choose our second direction to lie orthogonal to the rotation plane e.g. x^5. We then find

$$\bar{F}^{-1} \equiv 1 + \sum_{n=-\infty}^{\infty} [\breve{F}^{-1}(t, x^1, x^2, x^3, x^4, x^5 - n) - 1] \simeq 1 + \frac{2\pi m_0}{\bar{R}}\,, \tag{27}$$

$$\bar{\omega}_1 \equiv \sum_{n=-\infty}^{\infty} \breve{\omega}_1(t, x^1, x^2, x^3, x^4, x^5 - n) \simeq -\frac{2\pi m_0 a}{3\bar{R}^3} x^2\,, \qquad \bar{\omega}_2 \simeq 0\,, \tag{28}$$

where

$$\bar{R}^2 \equiv \sum_{i=2}^{4} (x^i)^2\,. \tag{29}$$

Asymptotically, the resulting geometry is given by:

$$\begin{aligned} ds^2 &= -\left(1 + \frac{2\pi m_0}{\bar{R}}\right)^{-2} [dt - \frac{2\pi m_0 a}{3\bar{R}^3} x^2 dx^1]^2 + \sum_{i=1}^{4} (dx^i)^2 \\ &= -\left(1 + \frac{2\pi m_0}{\bar{R}}\right)^{-2} dt^2 + \frac{4\pi m_0 a x^2}{3\bar{R}^3 \left(1 + \frac{2\pi m_0}{\bar{R}}\right)^2} dt dx^1 \\ &\quad + \left(1 - \frac{4\pi^2 m_0^2 a^2 (x^2)^2}{9\bar{R}^6 \left(1 + \frac{2\pi m_0}{\bar{R}}\right)^2}\right) (dx^1)^2 + \sum_{i=2}^{4} (dx^i)^2. \end{aligned} \tag{30}$$

Since x^1 is now a compact direction, we can extract the four dimensional fields from this five dimensional metric using the Kaluza-Klein decomposition

$$ds^2 = [g_{\mu\nu}(x) + A_\mu(x) A_\nu(x)] dx^\mu dx^\nu + 2A_\mu dx^\mu dy + e^{2\sigma} dy^2\,, \tag{31}$$

where $\mu = 0, 2, 3, 4$ and $y = x^1$. The result is

$$e^{2\sigma} = 1 - \frac{4\pi^2 m_0^2 a^2 (x^2)^2}{9\bar{R}^6 \left(1 + \frac{2\pi m_0}{\bar{R}}\right)^2} \simeq 1 \;\;, \tag{32}$$

$$A_t = \frac{2\pi m_0 a x^2}{3\bar{R}^3 \left(1 + \frac{2\pi m_0}{\bar{R}}\right)^2} \simeq \frac{2\pi m_0 a}{3\bar{R}^3} x^2 \;, \tag{33}$$

$$A_i = 0 \qquad \text{for} \qquad i = 2, 3, 4 \tag{34}$$

$$g_{tt} = -\left(1+\frac{2\pi m_0}{\bar{R}}\right)^{-2}\left(1+\frac{4\pi^2 m_0^2 a^2 (x^2)^2}{9\bar{R}^6\left(1+\frac{2\pi m_0}{\bar{R}}\right)^2}\right) \simeq -\left(1+\frac{2\pi m_0}{\bar{R}}\right)^{-2}\,, \tag{35}$$

$$g_{ti} = 0, \qquad g_{ik} = \delta_{ik} \qquad \text{for} \qquad i,k = 2,3,4\,. \tag{36}$$

where we have kept only the leading order terms on the right. Notice that the angular momentum is zero, but the gauge field A_μ clearly has a nonzero electric dipole moment which is proportional to a, and pointing in the x^2 direction. Reducing the antisymmetric tensor field yields another $U(1)$ gauge field in four dimensions $B_{\mu 1}$ with exactly the same dipole moment

$$B_{t1} = -\bar{F}\bar{\omega}_1 \simeq \frac{2\pi m_0 a}{3\bar{R}^3}x^2\,, \tag{37}$$

$$B_{i1} = 0 \qquad \text{for} \quad i = 2,3,4\,. \tag{38}$$

The four dimensional antisymmetric tensor field vanishes $B_{ti} = B_{ik} = 0$. Finally, the gauge fields that were already present in six dimensions reduce to give four dimensional gauge fields

$$A_t^{(j)} = \frac{p^{(j)}}{\sqrt{2}}\left[\left(1+\frac{2\pi m_0}{\bar{R}}\right)^{-1} - 1\right]\,, \tag{39}$$

$$A_i^{(j)} = 0 \qquad \text{for} \quad i = 2,3,4 \qquad (j = 1,\ldots,4)\,, \tag{40}$$

as well as nonzero scalars

$$A_1^{(j)} = \frac{p^{(j)}}{\sqrt{2}}\bar{F}\bar{\omega}_1 \simeq -\frac{\sqrt{2}\pi m_0 a}{3\bar{R}^3}p^{(j)}x^2 \qquad (j = 1,\ldots,4)\,. \tag{41}$$

From (35) and (39) it is clear that the mass and charge of this solution is independent of the value of the electric dipole moment. Therefore in four dimensions the resulting geometry is that of the extremally charged dilatonic black hole with an additional dipole moment in the x^2 direction. Higher order terms in the expansion of F and ω_i will now yield higher order multipole moments as well, but their values are determined in terms of m_0 and a.

There is increasing evidence that massive BPS string states are described by extreme black holes. Are there elementary string states which correspond to these black holes with electric dipole moment? If the dipole moment is not too large, the answer is yes. This follows from [15] where the fields outside of certain macroscopic string sources wrapped around a compact direction in D dimensions were constructed and shown to reduce to rotating black holes in $D-1$ dimensions. (These solutions were also discussed in [16].) Consider a string in six dimensions that reduces to a black hole rotating in the x^1, x^2 plane in five dimensions. Since these fields can also be superposed, one can compactify the x^1 direction by taking an infinite chain of such strings. One can show that the resulting four dimensional solution again describes a nonrotating black hole with electric dipole moment. Since the angular momentum associated with the string source satisfies the Regge bound, the parameter a is bounded, and the dipole moment cannot take arbitrarily large values.

To summarize, we have studied extreme black holes in heterotic string theory which are rotating in a compactified direction. From the four dimensional viewpoint, these solutions describe nonrotating black holes with electric dipole moments. Although we have discussed only electric charges and dipole moments, one can clearly perform an S-duality transformation on the four dimensional fields to obtain magnetic charges and dipole moments. Are more general solutions possible with independent higher multipole moments? (See [17] for an interesting example of nonspherical black holes in a different context.) Since the extreme black hole has a singularity at the origin and no real event horizon, it might appear easy to construct a large class of similar static solutions with $M^2 = Q^2/2$ and arbitrary multipole moments. However, in general these solutions will have timelike singularities at the origin rather than the more mild null singularities found here. Furthermore, they are unlikely to be supersymmetric. The solution constructed here has unbroken supersymmetry (at least to leading order in α') since it arises from the dimensional reduction of a chiral null model [8].

It has been shown that black holes with both electric and magnetic charges can have nonzero horizon area even in the extremal limit [18,19]. It would be interesting to construct

the rotating analog of these solutions in higher dimensions and compactify one direction in the plane of rotation. The result would appear to be a nonrotating black hole with dipole moment and nonzero horizon area. This seems to contradict the uniqueness theorems. (Although a theorem has not yet been rigorously established for this case, it is widely believed to hold.) The resolution of this apparent contradiction will certainly deepen our understanding of the properties of black holes.

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